3. Mr. Morgan's paper concerned the history and construction of five of the earlier logarithmic tables, namely, Henry Briggs, *Arithmetica Logarithmica*, London, 1624; Adrian Vlacq, *Arithmetica Logarithmica*, Goudae, 1628; Briggs-Gellibrand, *Trigonometria Britannica*, Goudae, 1633;. Dodson, *Anti-Logarithmic Canon*, London, 1742, and Vega, *Thesaurus Logarithmorum*, Leipzig, 1794. Copies of each of these works which are now all entirely out of print and very scarce were exhibited at the meeting. The *Arithmetica Logarithmica* of Briggs and the *Trigonometria Britannica* are the only fourteen place tables ever published. The *Anti-Logarithmic Canon* is an eleven place table of anti-logarithms, the most extensive ever constructed. The *Arithmetica Logarithmica* of Vlacq and the *Thesaurus* of Vega are both ten place tables. The latter gives also the Wolframmi table of Naperian logarithms to forty-eight decimal places.

4. The nature of the conic generated by projective ranges and pencils, as determined by the projective relation involved in its generation, has been obtained by Steiner by the use of synthetic methods. In this paper Miss Carlson obtained the same and other results by the use of analytic geometry methods.

5. In his paper Professor Brink explained the purpose of nomograms and the method of using them. He exhibited nomograms for the solution of right triangles and other problems, and indicated their usefulness in teaching college mathematics.

6. Professor Reuterdahl discussed the significance of Space and Time for the Finite and Infinite of mathematics. Space and Time may be regarded as the cosmic background of all knowledge. The finite and infinite are inseparable by-products of Space-Time. As a dual cosmic principle, Space-Time is a dynamic totality. Time is the space-separator and space is the time-binder. The onward urge of time conveys the impression of incompleteness and thus time constitutes the dynamic element of the dual principle Space-Time. Time is like a shearing blade which dissects space. Time can shear space only in three distinct ways and therefore physical space is three dimensional. Spatial manifolds of higher dimensionality are merely conceptual spatial extensions. Space-Time constitutes the only true continuum in the cosmos. The study of the finite and the infinite, consequently, becomes an investigation of the nature of this dynamic continuum. The paradox of a process-infinite composed of continuously associated finite phases finds its solution in the dynamic continuum of Space-Time.

GLADYS GIBBENS, *Acting Secretary.*

A SIMPLE FORM OF DUHAMEL'S THEOREM AND SOME NEW APPLICATIONS.[1]

By H. J. ETTLINGER, University of Texas.

1. Introduction. It is an open question whether Duhamel's[2] theorem should be given to a beginning class in the calculus. That one can avoid the use of this

[1] Read before the Mathematical Association of America, September, 1921.

[2] Duhamel, *Eléments de calcul infinitésimal*, Paris, 1856, p. 35.

theorem in dealing with *some* problems involving definite integrals such as length, volume, pressure, etc., must be recognized.[1] On the other hand, Duhamels' theorem is such a powerful tool for use in evaluating the limit of a sum by means of a definite integral and provides such a simple test as to when one infinitesimal may be replaced by another of the same order in this type of problem, that it is highly desirable to have it among the elementary theorems of the worker in mathematics. The theorem, furthermore, has important applications to the transformation of a double integral and the solution of integral equations.

W. F. Osgood[2] was the first to call attention to a rigorous formulation of the theorem. His proof makes use of uniform convergence to a limit and does not readily adapt itself to elementary treatment. The proof itself, based on Abel's lemma, is rather direct and simple,[3] though it is generally accepted that a proof involving uniform convergence should be taboo in a first course in calculus. However direct the proof may be, the *application* of the theorem is too difficult for a beginner and is not attempted in this form by Osgood in his text-book.[4]

R. L. Moore[5] has given a very general form of Duhamel's theorem whose importance, because of its generality, has been entirely overlooked. Some of the later writers simply mention Moore's form in passing. G. A. Bliss[6] presented a theorem equivalent to Duhamel's but involving the concept of uniform continuity. Recently E. V. Huntington[7] and H. B. Fine[8] presented simplified forms of Bliss's substitute. Finally G. James[9] has added a substitute different in form from the others.

It is the object of this paper to present Duhamel's theorem by means of a geometric lemma, which has the following advantages: (a) it is so simple in form that the beginning student in calculus will, from intuition, accept it as correct, subject to the statement that it will be proved in a later course; (b) it avoids the question of double limits and its concomitant concept, uniform convergence;[10]

[1] Cf. a suggestion by one who has "suffered" from the effects of Duhamel's theorem, Benjamin Graham, "Some calculus suggestions by a student," in this MONTHLY, *1917*, pp. 265–271.

[2] "The integral as the limit of a sum, and a theorem of Duhamel's," *Annals of Mathematics*, second series, vol. 4, 1903, pp. 161–178.

[3] Cf. W. F. Osgood, *A First Course in the Differential and Integral Calculus*, New York, 1907, pp. 164–165. The uniformity condition in the statement of the theorem is here omitted.

[4] *Ibid.*, pp. 166, 167, 168, 171, 173, 178, 182. In the applications no test of uniform convergence is made.

[5] "On Duhamel's theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. This paper, though mentioned by Bliss, Huntington and James, does not seem to have attracted the attention it deserves.

[6] "A substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 16, 1914, pp. 45–49.

[7] "On setting up a definite integral without the use of Duhamel's theorem," this MONTHLY, *1917*, pp. 271–275.

[8] "Note on a substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 19, 1918, pp. 172–173.

[9] "A substitute for Duhamel's theorem," *Tôhoku Mathematical Journal*, vol. 17, 1920, pp. 7–9.

[10] The secretary's report of the Wellesley meeting records an observation having been made indicating doubt as to the truth of this statement (see *1921*, 359–360). Responsibility for the origin of this observation could not be fixed. That there is no basis for this doubt is evident

(c) it is more general than the other forms;[1] (d) it is easily applied. In 2 the various forms of Duhamel's theorem are set forth for comparison. In 3 the geometric lemma is stated and Duhamel's theorem derived from it. In 4 an application is made to prove the existence of a definite integral for a continuous function and an example worked out for the usual class of problems in definite integrals. In 5 indications are given of applications of a more general form of the theorem to the transformation of a double integral and the solution of integral equations.

2. Various Forms of the Theorem. Osgood's[2] form of Duhamel's theorem is the following:

"*Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{A}$$

be a sum of infinitesimals and let α_i differ uniformly by an infinitesimal of higher order than Δx_i from the summand $f(x_i)\Delta x_i$ of the definite integral

$$\int_a^b f(x)dx \tag{B}$$

of the function $f(x)$, this function being continuous throughout the interval $a \leqq x \leqq b$. Then the sum (A) approaches a limit when $n = \infty$, and the value of this limit is the definite integral (B):

$$\lim_{n=\infty} \sum_{i=1}^{n} \alpha_i = \int_a^b f(x)dx."$$

The application of this theorem to any problem would call for a proof that

$$\lim_{n=\infty} \left[\frac{\alpha_k}{\Delta x} - f(x_k) \right] = 0$$

uniformly with respect to k. This is not to be done with ease in particular examples.

The following is Moore's[3] form:

"HYPOTHESIS: *(a) E is a limited point-set in a space of n-dimensions. $E_{1n}, E_{2n}, \cdots, E_{nn}$ are (for each value of the positive integer n) non-overlapping sub-sets of E of interior measures $e_{1n}, e_{2n}, \cdots, e_{nn}$, respectively. r_{in}, r_{in}' $(i = 1, \cdots, n)$ are numbers such that the set $\{|r_{in}' - r_{in}|\}$ is a bounded set, i.e., there exists a number c such that for all values of n and i $(i \leqq n)$, $|r_{in}' - r_{in}| \leqq c$. (b) $\operatorname{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$ exists. (c) E_0 is a subset of E of measure 0. (d) If P is a point of E not belonging to E_0, then*

$$\operatorname{Lim}_{n=\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0.$$

upon reading the proofs of the lemma. Also compare Osgood's theorem on non-uniformly convergent series, p. 244, footnote, and Moore's theorem, p. 241, referred to in this paper.

[1] Moore's form, of which the lemma is a corollary, is, of course, to be excepted.

[2] See *Annals* paper referred to above, p. 173.

[3] *L.c.*, pp. 162–163.

CONCLUSION: $\lim\limits_{n=\infty} \sum\limits_{i=1}^{n} r_{in}'e_{in}$ *exists and equals*

$$\lim_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}.\text{''}$$

That this theorem is more general than Osgood's form has been shown by an independence example by Moore.[1] Unfortunately the extreme generality and nomenclature of Moore's theorem has caused its importance to be overlooked.

The substitute theorem of Bliss[2] is the following:

"*Let us consider a function of the form $f(p, p', p'')$ where p is a symbol for a set of values (x, y, z) and p' and p'' for analogous sets. The points p, p', p'' are to range over a closed measurable region V in xyz-space in which f is continuous and hence uniformly continuous.*

"*If the region V is divided into measurable sub-regions with maximum diameters less than δ and with volumes denoted by ΔV_k $(k = 1, 2, \cdots, n)$, and if in each region three points p_k, p_k', p_k'' are chosen, then*

$$\lim_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k', p_k'')\Delta V_k = \lim_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k, p_k)\Delta V_k = \int_V f(p, p, p)dV.\text{''}$$

The proof depends on the fact that f is uniformly continuous, a concept which Bliss frankly admits has little meaning to the average sophomore, but which he hopes will "leave a framework of proof in the mind of the student."

Huntington[3] simplifies Bliss's theorem in the following form:

"*Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in such a way that we are led to divide the interval into n small parts or 'elements,' Δx, and to regard P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $F(x), f(x), \cdots$, can be found, such that, no matter what value of x is considered, and no matter how small Δx may be, the contribution from a typical element, $x = x$ to $x = x + \Delta x$, can be expressed 'approximately' (see note 1) in the form*

$$[F(x)f(x) \cdots]\Delta x.$$

Then the required quantity P will be correctly given by the value of the definite integral

$$P = \int_a^b [F(x)f(x) \cdots]dx,$$

whenever the functions $F(x), f(x), \cdots$ are continuous from $x = a$ to $x = b$.

"*Note 1.* The word 'approximately' is here used in a technical sense, meaning that the exact value of the contribution in question lies between $[\overline{F}\cdot\overline{f} \cdots]\Delta x$ and $[\overline{\overline{F}}\cdot\overline{\overline{f}} \cdots]\Delta x$, where $\overline{F}, \overline{f}, \cdots$ are the smallest and $\overline{\overline{F}}, \overline{\overline{f}}, \cdots$ the largest values of $F(x), f(x), \cdots$ in the element."

Fine[4] also has given a simplification of Bliss's substitute. His statement is as follows:

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[1] *L.c.*, p. 166.

[2] *L.c.*, p. 46.

[3] *L.c.*, pp. 273 and 274. The proof as given in the footnote 1 on p. 273 is different from Bliss's.

[4] *L.c.*, p. 172.

"*Let $f_1(x)$, $f_2(x)$, $\cdots$, $f_p(x)$ denote any set of functions of x, finite in number, which are continuous in the interval (ab) and let*

$$F(x) = f_1(x)f_2(x) \cdots f_p(x).$$

"*Suppose the interval (ab) to be divided and redivided into parts in any manner such that as the process is indefinitely continued the greatest of the parts will approach 0 as a limit, and at any stage in the process let h_1, h_2, $\cdots$, h_n represent the parts in length and position; also let ξ_i', ξ_i'', $\cdots$, $\xi_i^{(p)}$ and ξ_i denote any numbers in the part h_i $(i = 1, 2, \cdots, n)$. Then*

$$\lim_{n \doteq \infty} \sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'') \cdots f_p(\xi_i^{(p)})h_i = \lim_{n \doteq \infty} \sum_{i=1}^{n} F(\xi_i)h_i = \int_a^b F(x)dx."$$

The proof[1] of this theorem, like the proof of Bliss's, is based on uniform continuity.

James[2] gives the following substitute:

"*Divide the interval, a to b, into n positive subintervals, Δx_i, $i = 1, 2, 3, \cdots, n$, each of which converges to zero as n increases. If*

$$\lim_{n \to \infty} \sum_{1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx,$$

then

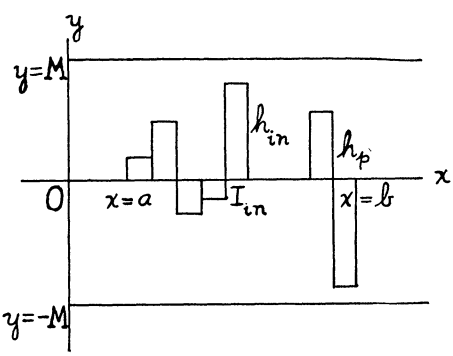
$$\lim_{n \to \infty} \sum_{1}^{n} [f(x_i)\Delta x_i + \phi_i(n)\Delta x_i]$$

exists and is equal to

$$\int_a^b f(x)dx,$$

provided there exists a sequence of constants c_1, c_2, c_3, $\cdots$ (independent of n),[3] with limit zero, such that $|\phi_i(n)| \leqq c_i$, $i = 1, 2, \cdots n$."

This form though more general than Osgood's is less general than Moore's, and the simplicity of the proof and ease of application are debatable.



3. A Geometric Lemma and a Simple Form of Duhamel's Theorem. The following lemma is stated in geometric form because it is believed that it will appeal to the student's intuition and will be one which he will readily accept, subject to the remark that the proof must be deferred until a later time.

LEMMA: — *Let the interval $I : a \leqq x \leqq b$ be divided into n equal closed subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base*

[1] *L.c.*, p. 173.

[2] *L.c.*, pp. 7–8.

[3] Parenthesis inserted by the present writer.

draw a rectangle of area, R_{in}, and height, h_{in}, such that

$$|h_{in}| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle whose base contains $\overline{P}$. If for each fixed $\overline{P}$ the altitude of this rectangle, $h_{\overline{P}}$, approaches 0 as a limit as $n \rightarrow \infty$, $\lim_{n\rightarrow\infty} h_{\overline{P}} = 0$, then $\lim_{n\rightarrow\infty} \sum_{i=1}^{n} R_{in} = 0$.

The proof[1] of this lemma is omitted, because it has no immediate relation to the purpose of this paper, but references to proofs are given in the footnote. The acceptance of the above lemma as true may be compared to the acceptance of Rolle's theorem as true from geometric intuition, or the acceptance of the following theorem: *If $f(x)$ is continuous in $a \leqq x \leqq b$ and $f(a) \neq f(b)$, then the equation $f(x) = N$, where N is between $f(a)$ and $f(b)$, has at least one root in $a < x < b$.* In fact Huntington[2] explicitly assumes this latter theorem without proof, and Graham,[3] who also uses it, refers to Goursat's *Cours d'Analyse Mathématique* for a proof.

On the basis of this geometric lemma we can readily establish

DUHAMEL'S THEOREM:—*Let the interval $I : a \leqq x \leqq b$ be divided into n equal subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw two rectangles, R_{in}' and R_{in}'', of height h_{in}' and h_{in}'', respectively, such that*

$$|h_{in}' - h_{in}''| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant, independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle, R_{in}', whose base contains $\overline{P}$ and a corresponding rectangle, R_{in}'', with the same base. If, for each fixed $\overline{P}$, the difference of the altitudes of these rectangles, $h_{\overline{P}}'$ and $h_{\overline{P}}''$, approaches zero as a limit,

$$\lim_{n\rightarrow\infty} (h_{\overline{P}}' - h_{\overline{P}}'') = 0,$$

[1] A proof may be given similar to Moore's (*l.c.*, p. 163), based on the application of a theorem due to W. H. Young (*Proceedings of the London Mathematical Society*, series 2, vol. 2, 1905, p. 25). Still another proof is that of E. Landau (*Mathematische Zeitschrift*, vol. 2, 1918, pp. 350–351). Professor Osgood, in a recent letter, pointed out that the lemma is virtually contained in a theorem which he proved in the *Amer. Jour. of Math.*, vol. 19, 1897, p. 188. He showed that, *if $s_n(x)$ be continuous in the interval $a \leqq x \leqq b$ for all values of n, and if $s_n(x)$ approach a continuous limit; if, furthermore, $s_n(x)$, regarded as a function of the two independent variables x and n, remain finite, then*

$$\lim_{n=\infty} \int_a^b s_n(x)dx = \int_a^b \lim_{n=\infty} s_n(x)dx.$$

From this theorem the lemma follows at once. Let $s_n(x)$ be defined as follows: On I_{in} as base erect an isosceles triangle whose altitude is $2h_{in}$. The equal legs of these triangles shall form the graph of $s_n(x)$. And now the area under $s_n(x)$ is precisely $\sum_{i=1}^{n} R_{in}$. But $\lim_{n=\infty} s_n(x) = 0$. Hence $\lim_{n=\infty} \sum_{i=1}^{n} R_{in} = 0$.

Since this paper was written the author has given an elementary proof based on the usual fundamental limit theorems of the beginning of the calculus (presented to the American Mathematical Society, September 7, 1922).

[2] *L.c.*, p. 273, footnote 1. [3] *L.c.*, p. 268, Theorem 3, Lemma.

and if $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n$ *exists, then* $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n$ *exists and*

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n.$$

Let

$$h_{in} = h_{in}' - h_{in}''.$$

Then $|h_{in}| \leqq M$ for all values of n, and $\lim_{n\to\infty} h_{\bar{P}} = 0$ for every fixed P. Hence by the lemma,

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}\Delta x_n = 0 \qquad \text{or} \qquad \lim_{n\to\infty}\sum_{i=1}^{n} (h_{in}' - h_{in}'')\Delta x_n = 0.$$

Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i-1}^{n} h_{in}''\Delta x_n.$$

4. Applications to Definite Integrals. We proceed to use the theorem of 3 to show that if $f(x)$ is continuous in the interval $I : a \leqq x \leqq b$, then $f(x)$ is integrable in I. Let I be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b - a)/n$. Let H be the maximum value of $f(x)$ in I; and h the minimum value of $f(x)$ in I. Let $\bar{P}$ be any fixed point of I and let $I_{\bar{P}n}$, for each value of n, be a closed subinterval I_{in} which contains $\bar{P}$, Let $H_{\bar{P}n}$ be the largest value of $f(x)$ in $I_{\bar{P}n}$ and let $h_{\bar{P}n}$ be the smallest value of the set $H_{\bar{P}1}$, $H_{\bar{P}2}, \cdots, H_{Pn}$. Let h_{in} be defined for each I_{in} as the number which for each $\bar{P}$ in I_{in} is equal to the value of $h_{\bar{P}n}$ which corresponds to $\bar{P}$. For each point $\bar{P}$, h_{in} never increases as n increases. Hence $A_n = \sum_{i=1}^{n} h_{in}\Delta x_n$ never increases as n increases. But $A_n \geqq h(b - a)$. Therefore as $n \to \infty$, A_n approaches a limit or $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}\Delta x_n$ exists.

Let ξ_{in} be any value of x in I_{in} and $\bar{\xi}_{in}$, the value of ξ_{in} which corresponds to $I_{\bar{P}n}$. Now $\lim_{n\to\infty} h_{\bar{P}n} = f(\bar{x})$ where $\bar{x}$ is the abscissa of $\bar{P}$, and $\lim_{n\to\infty} f(\bar{\xi}_{in}) = f(\bar{x})$ since $f(x)$ is continuous in I. Also $|h_{in} - f(\xi_{in})| \leqq H - h$ for every value of $i(\leqq n)$ and n. Hence by the theorem of 3,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n$$

exists. This limit is independent of the choice of ξ_{in}, and of the method of subdivision.

Hence by the usual argument

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n = \int_a^b f(x)dx.$$

We may now restate Duhamel's theorem:

Let $f(x)$ be a continuous function of x in the interval $I : a \leqq x \leqq b$. Let I be subdivided into n equal closed subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw a rectangle of height β_{in}. Let ξ_{in} be any value of x in I_{in}. If $\bar{P}$ is any fixed point of I whose $x = \bar{x}$, there is for each value of n at least one rectangle with height, $\beta_{\bar{P}}$, whose base $\bar{I}_{in}$ contains $\bar{P}$. If $|\beta_{in}| \leqq M$ $(i = 1, 2, \cdots n)$ for all values of n, where M is a constant, independent of i and n, and if for each fixed point $\bar{P}$ $\lim_{n\to\infty} \beta_{\bar{P}} = f(\bar{x})$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \beta_{in}\Delta x_n = \int_a^b f(x)dx.$$

If we identify $f(\xi_{in}) = h_{in}'$ and $\beta_{in} = h_{in}''$ of the theorem in 3, the derivation of the preceding theorem is immediate.

For the application of this form of Duhamel's theorem to the usual problems of the first course of the calculus, it is only necessary to refer to the examples in Osgood's[1] text-book with some modifications. In each of the examples cited the theorem of 3 is tacitly assumed by Osgood and applied. For $x = x_k$ in these instances cannot be taken as designating the right hand end of the kth subdivision since $\lim_{n\to\infty} x_k = a$ and $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ (where $\beta_k = \beta_{kn}\Delta x_n$, $\alpha_k = f(x_{kn})\Delta x_n$ in our notation) become meaningless as far as the application of Osgood's form of the theorem is concerned. Professor Osgood has indicated in another letter that he regards x_k as fixed. This is impossible unless k varies with n and n increases without limit in some special manner. What is evidently intended is that x_k shall be regarded as fixed entirely independent of any change in n and, furthermore, that it shall represent any value of x in I, *i.e.*, x_k is the $\bar{x}$ of our theorem. It is also desirable to replace $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ by $\lim_{n\to\infty} (\beta_k - \alpha_k) = 0$, in our notation $\lim_{n\to\infty} (h_{\bar{P}}' - h_{\bar{P}}'') = 0$, since this avoids the difficulty of $\alpha_k = 0$. With these interpretations we reproduce the example on p. 166 in Osgood's text-book.

To find the length of the curve $y = F(x)$ from $x = a$ to $x = b$. For simplicity let $F(x)$ be continuous with a continuous derivative in $I : a \leqq x \leqq b$. Subdivide I into n equal parts; erect ordinates at the points of division; and inscribe a broken line in the arc to be measured. The length of this line is

$$\sum_{i=0}^{n-1} \sqrt{\Delta x_n^2 + \Delta y_{in}^2} = \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}\,\Delta x_n.$$

Now, to make connections with the foregoing theorem, let

$$f(x) = \sqrt{1 + [F'(x)]^2}$$

and

$$\beta_{in} = \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}\,.$$

[1] *L.c.*, see examples on pp. 166, 167, 168, 171, 173, 178, 182.

Now $|\beta_{in}| = \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}$ remains finite and $\lim_{n \to \infty} \beta_{\bar{P}} = \sqrt{1 + [F'(\bar{x})]^2} = f(\bar{x})$. Hence the conditions of Duhamel's theorem are satisfied and

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\Delta x_n^2 + \Delta y_n^2} = \int_a^b \sqrt{1 + [F'(x)]^2}\, dx.$$

It seems to the writer that the form of Duhamel's theorem here given, on account of its simplicity and ease of application, should earn a place in our elementary calculus texts.[1]

5. Applications to the Transformation of a Double Integral and the Solution of an Integral Equation.[2] The geometric lemma and the statement of Duhamel's theorem of 3 may be generalized in several directions. It is quite obvious that it is not essential that the lengths of the subdivisions be equal, and instead of a linear interval we may have a closed space of any number of dimensions. These indications are merely special cases of Moore's general form which was reproduced in 2. We proceed to apply this form to the transformation of a double integral.[3]

HYPOTHESES: 1. *$f(x, y)$ is a function having an upper and lower bound defined for all points (x, y) of a closed, connected domain, G, contained in H, a bounded connected domain, the boundary of G having content zero;*

2. *x and y are expressed in terms of ξ and η by the relations*

$$x = f_1(\xi, \eta), \qquad y = f_2(\xi, \eta), \tag{T}$$

such that the transformation (T), as well as its inverse

$$\xi = \phi_1(x, y), \qquad \eta = \phi_2(x, y), \tag{T'}$$

is continuous and one-to-one. Let $F(\xi, \eta) \equiv f[f_1(\xi, \eta), f_2(\xi, \eta)]$ and let G' in H' represent the transform of G in H. L is a set of points of G of measure zero;

3. *$f(x, y)$ is continuous in G except at points of L;*

4. *the partial derivatives $\frac{\partial f_1}{\partial \xi}, \frac{\partial f_1}{\partial \eta}, \frac{\partial f_2}{\partial \xi}, \frac{\partial f_2}{\partial \eta}$ exist and are continuous at every point of H', except at points of the set L' corresponding to L;*

5. *the Jacobian of $f_1(\xi, \eta)$, $f_2(\xi, \eta)$ with respect to ξ and η, $J = \frac{\partial(f_1, f_2)}{\partial(\xi, \eta)}$,* is *different from zero in H', except at points of L'.*

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[1] The ordinary "carefree" treatment of Duhamel's theorem may be exemplified in an otherwise most excellent text, G. A. Gibson, *Elementary Treatise on the Calculus*, London, 1919. In many respects this book is superior to the average calculus text-book. On p. 198 the author states that Duhamel's theorem "is not necessarily true if the infinitesimals are not all of the same sign." That this is incorrect is established by the present paper. He adds that "from its use in integration this theorem is often called the fundamental theorem of the integral calculus."

[2] Since this paper was written, the author has made other important applications (1) to the transformation of a simple integral, (2) to the differentiation or integration termwise of a non-uniformly convergent series, (3) (by a pupil) to the approximate solution of differential equations, (4) to Green's theorem in potential theory and related problems.

[3] See E. W. Hobson, *The Theory of Functions of a Real Variable*, Cambridge, 1907, pp. 445–452.

CONCLUSION:

$$\int\int_G f(x, y)dxdy = \int\int_{G'} F(\xi, \eta)\left|\frac{\partial(f_1, f_2)}{\partial(\xi, \eta)}\right| d\xi d\eta.$$

The outline of the proof is as follows:

$$\begin{aligned}\int\int_G f(x, y)dxdy &= \lim_{n\to\infty}\sum_{i=1}^{n^2} f(x_{in}, y_{in})\Delta G_{in} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n^2} f[f_1(\xi_{in}, \eta_{in}), f_2(\xi_{in}, \eta_{in})]\Delta G_{in} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n^2} F(\xi_{in}, \eta_{in})\frac{\Delta G_{in}}{\Delta G_{in}'}\cdot\Delta G_{in}'.\end{aligned}$$

If $\overline{P}(\bar{x}, \bar{y})$ is a fixed point of G not in L, then for each value of n there is (at least) one $\Delta\overline{G}_{in}$ containing $\overline{P}$. If $\Delta\overline{G}_{in}'$ is the transform of $\Delta\overline{G}_{in}$ by T, then it may be proved that

$$\lim_{n\to\infty}\left[\frac{\Delta\overline{G}_{in}}{\Delta\overline{G}_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|}\right] = 0^1 \qquad \text{and} \qquad \left|\frac{\Delta G_{in}}{\Delta G_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|}\right| \leqq M$$

for each value of n.

Hence by Moore's form of Duhamel's theorem the transformation is established.

Fredholm[2] solved the integral equation of the second kind,

$$u(x) = f(x) + \int_a^b K(x, \xi)u(\xi)d\xi, \tag{E}$$

by means of a system of algebraic equations in an infinite number of unknowns. We give Bôcher's[3] treatment:

Let the interval $I : a \leqq x \leqq b$ be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b - a)/n$ and let $x_{0n} = a$, x_{1n}, x_{2n}, $\cdots$, $x_{nn} = b$ designate the points of division. If we replace (E) by the equation

$$u(x) = f(x) + \sum_{j=1}^{n} K(x, x_{jn})u_n(x_{jn})\Delta x_n, \tag{E'}$$

(E') is to be satisfied for the values x_{in} $(i = 1, 2, \cdots n)$. This yields the system of n equations

$$u_n(x_{in}) = f(x_{in}) + \sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n \quad (i = 1, 2, \cdots n) \tag{S}$$

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1 Cf. Goursat, *Cours d'Analyse Mathématique*, Paris, 1902, vol. 1, pp. 300–302.

2 "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kungl. Vetenskaps-Akademiens Förhandlingar*, vol. 57, 1900, p. 39. Also *Acta Mathematica*, vol. 27, 1903, pp. 365–390.

3 *Cambridge Tracts in Mathematics and Mathematical Physics*, no. 10, *An Introduction to the Study of Integral Equations*, Cambridge, at the University Press, 1909, pp. 25–27.

in n unknown quantities, $u_n(x_{in})$, $(i = 1, 2, \cdots n)$. (S) may be written as

$$-\sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n + u_n(x_{in}) = f(x_{in}) \quad (i = 1, 2, \cdots n). \qquad (S')$$

By Cramer's formula the value of $u_n(x_{\mu n})$ is

$$u_n(x_{\mu n}) = \frac{1}{D_n}\sum_{i=1}^{n} f(x_{in})D_n(x_{\mu n}, x_{in}) \quad (\mu = 1, 2, \cdots n),$$

where $D_n(\neq 0)$ is the determinant of the system [1] and $D_n(x_{\mu n}, x_{\nu n})$ is the cofactor of the element in the νth row and the μth column. The solution of (E) is now obtained by letting n become infinite and at the same time allowing μ to vary with n in such a manner that $x_{\mu n}$ approaches a fixed value x in I. If $D = \lim_{n\to\infty} D_n$ and $D(x, \xi) = \lim_{n\to\infty} \frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, then it may be proved that

$$u(x) = f(x) + \frac{1}{D}\int_a^b f(\xi)D(x, \xi)d\xi.$$

This is the solution of (E).

The application of Duhamel's theorem in this proof is that of determining the limiting form of $\frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$. The hypothesis [2] concerning $K(x, \xi)$ is that it shall be bounded in $S : a \leqq x \leqq b,\ a \leqq \xi \leqq b$ and continuous except for a set of points, L, of measure zero, and $K(x, x)$ shall be integrable in $I : a \leqq x \leqq b$. The typical term [3] of $\frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, except for the omission of the numerical coefficient, is

$$\sum_{\underbrace{i, j, \cdots}_{m-1} = 1}^{n} \begin{vmatrix} K(x_{\mu n}, x_{\nu n}) & K(x_{\mu n}, x_{in}) & K(x_{\mu n}, x_{jn}) & \cdots m \text{ columns} \\ K(x_{in}, x_{\nu n}) & K(x_{in}, x_{in}) & K(x_{in}, x_{jn}) & \cdot \\ K(x_{jn}, x_{\nu n}) & K(x_{jn}, x_{in}) & K(x_{jn}, x_{jn}) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} \ldots & \ldots & \ldots & \cdot \end{vmatrix} \overline{\Delta x}_n^{m-1}. \qquad (A)$$

If (x, ξ) is a fixed point in S not in L such that as n increases $x_{\mu n}$ approaches x and $x_{\nu n}$ approaches ξ, then the limit of (A) would appear to be

$$\underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{m-1} \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & K(x, \xi_2) & \cdots m \text{ columns} \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & \cdot \\ K(\xi_2, \xi) & K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} \ldots & \ldots & \ldots & \cdot \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_{m-1}. \qquad (B)$$

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[1] Bôcher, *l.c.*, p. 26.
[2] *L.c.*, p. 36, theorem 3.
[3] *L.c.*, p. 27.

That this is the limit can be proved by identifying (A) above with $\Sigma r_{in}'e_{in}$ and (B) with $\lim\limits_{n\to\infty} \Sigma r_{in}e_{in}$ of the theorem of p. 241.

Now $K(x, \xi)$ is bounded; hence

$$|r_{in}' - r_{in}| \leqq C$$

and $\lim\limits_{n\to\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0$, if P is not a point of L, since $K(x, \xi)$ is continuous except for points of L. Hence Duhamel's theorem establishes the limiting form of (A) to be (B).

NOTE ON APPLICATION OF DIOPHANTINE ANALYSIS TO GEOMETRY.

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It frequently becomes desirable, in teaching analytic geometry of three dimensions, to know a set of three mutually perpendicular lines each of which has all its direction cosines rational. It is well known that if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively, the direction cosines of their common perpendicular are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, and $(l_1m_2 - l_2m_1)$, and that if the two first-mentioned lines are perpendicular the factor of proportionality is unity. Hence, if two of our three mutually perpendicular lines have rational direction cosines, the third will also have this property. Furthermore, it will presently appear that one of these lines can be taken to be any line having rational direction cosines.

We therefore have first to find rational solutions of the equation

$$l_1^2 + m_1^2 + n_1^2 = 1. \tag{1}$$

It is well known, and can easily be verified, that all such solutions are given by the formulæ

$$l_1 = \frac{p^2 + q^2 - r^2}{p^2 + q^2 + r^2}, \qquad m_1 = \frac{2pr}{p^2 + q^2 + r^2}, \qquad n_1 = \frac{2qr}{p^2 + q^2 + r^2},$$

where p, q, and r are any three integers.

Having so selected l_1, m_1, and n_1, we have next to solve, in rational numbers, the simultaneous equations

$$\begin{cases} l_2^2 + m_2^2 + n_2^2 = 1, \\ l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{cases} \tag{2}$$

If we eliminate l_2 from these equations, the resulting equation can be put into the form

$$\left\{m_2 + \frac{m_1n_1n_2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1n_2}{l_1^2 + m_1^2}\right\}^2 = \left\{\frac{l_1^2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1m_1}{l_1^2 + m_1^2}\right\}^2, \tag{3}$$

since the second member may be written as $l_1^2/(l_1^2 + m_1^2)$. Two solutions of